International Journal of Theoretical Physics, Vol. 15, No. 5 (1976), pp. 377–387

Some Structurological Remarks on a Nonlocal Field

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Received: 7 *July* 1975

Abstract

In this paper, continued from the last paper (Ikeda, 1974), two kinds of structurological generalizations of our nonlocal field (i.e., the (x, ψ) field) are considered physicogeometrically. One is a Finslerian generalization, where the base field [i.e., the (x) field] is extended to a Finslerian field and Weyl's gauge field (i.e., the electromagnetic potential) is physically identified with the directional vector adopted as the internal variable in the ordinary nonlocal field theory. Another is a generalization by which the spinor (ψ) itself is taken as an independent variable, where some inherent characteristics of ψ are fused into the spatial structure. The latter is regarded as a "nonlocalization" of the (x) field accomplished by attaching ψ to each point, in the true sense of the word. Particularly, the spatial structures of these generalized nonlocal fields are described in detail.

1. *Introduction*

A certain kind of nonlocal field [viz., the (x, ψ) field] has been constructed in a previous paper (Ikeda, 1974) by attaching to each point (x) of the base field [viz., the (x) field] the four-component spinor (ψ) chosen as the internal freedom. This nonlocal field has also been regarded as an interaction field or a unified field between the (x) and (ψ) fields; the former has been likened to the gravitational field governed by general relativity, while the latter to the spinor field governed by quantum mechanics. And we may say, in a certain sense, that this field can be obtained by "nonlocalizing" the (x) field and that the (x) field is "nonlocalized" by attaching ψ to each point. However, in our theory (Ikeda, 1974), the spinor (ψ) itself is not adopted as the independent variable, so that in the strict sense of the word, the (x, ψ) field cannot be regarded as a nonlocal field obtained by "nonlocalizing" the (x) field. Therefore, we shall consider these circumstances in the following.

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In the ordinary nonlocal field theory (Yukawa, 1950), a certain kind of directional vector, say d, is taken as the internal variable associated with each point, and both point (x) and directional vector (d) are chosen as the independent variables of this field. Therefore, as pointed out by Horváth (1958) and Takano (1968), this field is geometrically grasped by Finsler geometry. From this, it is found that when the internal variable is a vectorial quantity and is also chosen as an independent variable, this kind of "nonlocalized" field can be considered geometrically by Finsler geometry under the premiss that each physical quantity is properly homogeneous with respect to the internal variable. Of course, our nonlocal field, namely, the (x, ψ) field, cannot be embedded in Finsler space, because the internal variable (ψ) is not vectorial. Therefore, from a geometrical point of view, the (x, ψ) field should be considered a nonlocal field in the higher-order space of order $M \geqq 2$ (or the Kawaguchi space of order $M \geqq 2$; $M = 1, 2, 3, \ldots$), in which an arc length along a curve $x^\kappa = x^\kappa(t)$ ($\kappa = 1, 2, 3, \ldots, n; n = 4$) is given by the integral

$$s = \int_{t_1}^{t_2} F(x^\kappa, x^{(1)\kappa}, \ldots, x^{(M)\kappa})\, dt \tag{1.1}$$

where $x^{(\alpha)\kappa} = d^\alpha x^\kappa/dt^\alpha$ ($\alpha = 1, 2, \ldots, M$), and t is an arbitrary parameter. And F is called the fundamental function, which, of course, must satisfy some homogeneity conditions in order that the arc length may be invariant under any parameter transformation (Kawaguchi, 1962). It should be noticed here that Finsler space is the higher-order space of order 1.

Now, as has often been pointed out (Horváth, 1950, 1958; Takano, 1968), the physical meaning of the internal variable (d or $x^{(1)}$) in the Finslerian physical field is somewhat arbitrary. Accordingly, since the internal variable is vectorial, it may be compared quite physically and heuristically to the electromagnetic potential (ρ) in Weyl's style (i.e., Weyl's gauge field), which has been introduced into our field theory (Ikeda, 1974) through Weyl's gauge invariance (Weyl, 1918). Therefore, if the base field, i.e., the (x) field, is likened to the gravitational field and the (ρ) field is assumed to be constituted by the electromagnetic potential, then a certain kind of unified field [viz., the (x, ρ) field] between the gravitational field [i.e., the (x) field] and the electromagnetic field [i.e., the (ρ) field] can be considered by means of Finsler geometry, as has been actually done by several authors (Horváth, 1950; Horváth und Moór, 1952; Lichnerowicz, 1955; Tonnelat, 1965, 1966).

Conversely, in order to construct a nonlocal field (or a unified field) by taking the electromagnetic potential (ρ) as the independent internal variable annexed to each point (x) and to fuse the concept of Weyl's gauge invariance into its spatial structure, we cannot help considering such a Finslerian nonlocal field as the (x, ρ) field. On the other hand, in our nonlocal field theory (Ikeda, 1974), Weyl's gauge field (ρ) is introduced through Weyl's gauge invariance a posteriori. Therefore, the electromagnetic potential (ρ) is not fused into the spatial structure of the (x, ψ) field. This is the reason why we must consider from the standpoint of structurology a Finslerian generalization of the (x, ψ) field. And when we want to consider this generalization, we had better construct first such a Finslerian nonlocal field as the (x, ρ) field and consider secondly a further generalized nonlocal field by taking account of some "interactions" between the (x, ρ) and (ψ) fields, because the (x, ψ) field itself cannot be considered by Finsler geometry, as mentioned above. This Finslerian generalization will be described in Section 2. This generalized nonlocal field will be called the $[x, \rho]$ field in the following.

As mentioned above, if the spinor (ψ) attached to each point as the internal variable is chosen as the independent variable, then ψ takes formally the place of the directional vector appearing in the ordinary nonlocal field theory (Yukawa, 1950). By doing so, our nonlocal field, i.e., the (x, ψ) field can be generalized in a different way, and this generalization is regarded as a "nonlocalization" of the (x) field attained by attaching ψ to each point, in the true sense of the word. Of course, since the spinor is obviously nonvectorial, this generalized nonlocal field, which will be named the $[x, \psi]$ field in the following, cannot be grasped by Finsler geometry. Therefore, it should be investigated by means of the theory of higher-order spaces of order $M \geqq 2$. This generalization will be considered in Section 3.

Thus, in this paper, two kinds of structurological generalizations of the (x, ψ) field will be considered and particularly, the spatial structures of these generalized nonlocal fields (viz., the $[x, \rho]$ and $[x, \psi]$ fields) will be described in detail. Besides, some fundamental remarks on the newly introduced spin gauge fields will also be made.

2. *A Finslerian Generalization of the* (x, ψ) *Field*

First, we shall explain some basic concepts underlying the theory of Finsler spaces (Cartan, 1934; Rund, 1959), but we select only those that are indispensable to the discussion that follows. In Finsler space, as is obtained from (1.1), an arc length between two points $x_1 = x_1(t_1)$ and $x_2 = x_2(t_2)$ along a curve $x^\kappa = x^\kappa(t)$ $(\kappa = 1, 2, 3, 4)$ is given by

$$z = \int_{t_1}^{t_2} F(x^\kappa, \rho^\lambda)\, dt \tag{2.1}$$

where t is an arbitrary parameter, and x^κ denote the coordinates of position and $\rho^\lambda (\equiv dx^\lambda/dt)$ the components of directional vector (or the directional coordinates), the latter being regarded as the internal degrees of freedom attached to each point (x) from a physical point of view. $F(x, \rho)$ means the fundamental function, which is assumed to be positively homogeneous of degree 1 in the ρ. This assumption is necessary because the arc length must be invariant under any parameter transformation. Then, the metric tensor is defined by

$$g_{\lambda\kappa} = \frac{1}{2} \frac{\partial^2 F^2(x, \rho)}{\partial \rho^\lambda \partial \rho^\kappa} \tag{2.2}$$

which becomes positively homogeneous of degree 0 in the ρ. Next, the absolute differential of an arbitrary vector X^κ is given by

$$DX^\kappa = dX^\kappa + \Gamma^\kappa_{\mu\lambda} X^\lambda\, dx^\mu + C^\kappa_{\mu\lambda} X^\lambda d\rho^\mu \tag{2.3}$$

where $\Gamma^\kappa_{\mu\lambda}$ and $C^\kappa_{\mu\lambda}$ are the coefficients of connection. Since the length of X^κ is assumed to be invariant under parallel displacement, the metric condition $Dg_{\lambda\kappa} = 0$ holds, from which the following relations are obtained:

$$\begin{aligned} \partial_\mu g_{\lambda\kappa} &= 2\Gamma_{\mu(\lambda\kappa)} = \Gamma_{\mu\lambda\kappa} + \Gamma_{\mu\kappa\lambda} \\ \partial_{\dot\mu} g_{\lambda\kappa} &= 2C_{\mu(\lambda\kappa)} = C_{\mu\lambda\kappa} + C_{\mu\kappa\lambda} \end{aligned} \tag{2.4}$$

where

$$\Gamma_{\mu\lambda\kappa} = \Gamma^\nu_{\mu\lambda} g_{\nu\kappa},\ C_{\mu\lambda\kappa} = C^\nu_{\mu\lambda} g_{\nu\kappa},\ \partial_\mu = \frac{\partial}{\partial x^\mu},\ \partial_{\dot\mu} = \frac{\partial}{\partial \rho^\mu}$$

At this stage we shall assume, for the sake of simplicity, that

$$C_{\mu\lambda\kappa} = C_{\mu\kappa\lambda} = C_{\lambda\mu\kappa} \tag{2.5}$$

as in the ordinary theory of Finsler spaces (Cartan, 1934; Rund, 1959). Therefore, $C_{\mu\lambda\kappa}$ is determined by

$$C_{\mu\lambda\kappa} = \tfrac{1}{2} \partial_{\dot\mu} g_{\lambda\kappa} \tag{2.6}$$

which becomes positively homogeneous of degree (-1) in the ρ. Then, from (2.5) and (2.6), the following relation is obtained:

$$C_{\mu\lambda\kappa}\rho^\mu = C_{\mu\lambda\kappa}\rho^\lambda = C_{\mu\lambda\kappa}\rho^\kappa = 0 \tag{2.7}$$

In Finsler space, the so-called base connection of ρ, i.e., $\delta\rho$, which represents the inherent absolute differential of ρ and connects ρ (or $d\rho$) with x (or dx) geometrically, turns out to be equivalent to the absolute differential of ρ, i.e., $D\rho$ because of the relation (2.7) (Kawaguchi, 1962). Therefore, (2.3) is rewritten as

$$DX^\kappa = dX^\kappa + \overset{*}{\Gamma}{}^\kappa_{\mu\lambda} X^\lambda dx^\mu + C^\kappa_{\mu\lambda} X^\lambda \delta\rho^\mu \tag{2.8}$$

where

$$\overset{*}{\Gamma}{}^\kappa_{\mu\lambda} = \Gamma^\kappa_{\mu\lambda} - G_\mu{}^\nu C^\kappa_{\nu\lambda}, \qquad G_\mu{}^\nu \equiv \Gamma^\nu_{\mu\iota}\rho^\iota \tag{2.9}$$

From (2.8), the covariant derivatives of X^κ are defined by

$$\begin{aligned} \nabla_\mu X^\kappa &= \overset{*}{\partial}_\mu X^\kappa + \overset{*}{\Gamma}{}^\kappa_{\mu\lambda} X^\lambda \\ \nabla_{\dot\mu} X^\kappa &= \partial_{\dot\mu} X^\kappa + C^\kappa_{\mu\lambda} X^\lambda \end{aligned} \tag{2.10}$$

where $\overset{*}{\partial}_\mu = \partial_\mu - G_\mu{}^\nu \partial_{\dot\nu}$. Of course, $\nabla_\mu g_{\lambda\kappa} = 0$ and $\nabla_{\dot\mu} g_{\lambda\kappa} = 0$ hold. By the way, it should be noticed that, in the following, the internal variable (ρ) is physically identified with the electromagnetic potential (i.e., Weyl's gauge field) from the beginning and each physical quantity in the $[x, \rho]$ field is assumed to be properly homogeneous with respect to ρ.

Now, we shall revert to our nonlocal field, i.e., the (x, ψ) field. As was already mentioned in the previous paper (Ikeda, 1974), which will be cited as [I] in the following, the metric in the (x, ψ) field is given by

$$d\sigma\psi = g_\lambda dx^\lambda \psi \qquad (\lambda = 1, 2, 3, 4) \tag{2.11}$$

which has close analogy to the metric in "wave geometry" (Mimura and Takeno, 1962; Mimura *et al.*, 1967) {cf. (2.6) in [I]}. In (2.11), $d\sigma$ $(\equiv g_\lambda dx^\lambda)$ denotes the linearized arc length in the (x, ψ) field, and g_λ means the matrix metric, which is assumed to contain the Dirac matrix γ_λ {cf. (2.7) in [I]}. The metric of the type (2.11) represents the "interaction" between the (x) and (ψ) fields, and it is necessary for us to introduce the metric of this type in order to consider a Finslerian generalization of the (x, ψ) field along our own way of thinking. That is, in order to carry out our purpose, we must first obtain the linearized arc length in the (x, ρ) field, which, as mentioned in section 1, is a Finslerian nonlocal field obtained by attaching the electromagnetic potential (ρ) to each point (x), and we must secondly define the metric in the $[x, \rho]$ field by the similar form to (2.11). Concerning the former, it has been known (Lichnerowicz, 1955; Tonnelat, 1965, 1966) that the trajectories of the charged particles are the geodesics of Finsler space families characterized by the elementary interval (i.e., the linearized arc length)

$$dz = \sqrt{\gamma_{\lambda\kappa} dx^\kappa dx^\lambda} + k\rho_\mu dx^\mu \tag{2.12}$$

where k $(\equiv e/m$, where e is the charge and m is the mass) is a constant, by which every Finsler space is determined relative to every type of particle, and $\gamma_{\lambda\kappa}$ denotes the Riemannian metric tensor stipulated by the Dirac matrix γ_λ as follows: $\gamma_{\lambda\kappa} = \gamma_{(\lambda}\gamma_{\kappa)} = (\gamma_\lambda\gamma_\kappa + \gamma_\kappa\gamma_\lambda)/2$ {cf. (2.2) in [I]}. Therefore, the linearized arc length (dz) in the (x, ρ) field can be written as

$$dz = \alpha_\lambda dx^\lambda, \qquad \alpha_\lambda \equiv \gamma_\lambda + k\rho_\lambda I \tag{2.13}$$

where dz and α_λ correspond to $d\sigma$ and g_λ, respectively. Consequently, the metric in the $[x, \rho]$ field is defined by

$$dz\psi = \alpha_\lambda dx^\lambda \psi \tag{2.14}$$

Therefore, we can construct the $[x, \rho]$ field through the same procedure as has been employed to construct the (x, ψ) field in [I], but in the following, we shall only point out some essentially Finslerian features underlying the $[x, \rho]$ field.

First, since α_λ functions like the Dirac matrix γ_λ, the absolute differential of it is given by [cf. (2.3)]

$$D\alpha_\lambda = d\alpha_\lambda - \Gamma^\kappa_{\mu\lambda}\alpha_\kappa dx^\mu - A^\kappa_{\mu\lambda}\alpha_\kappa dx^\mu - C^\kappa_{\mu\lambda}\alpha_\kappa d\rho^\mu - B^\kappa_{\mu\lambda}\alpha_\kappa d\rho^\mu \tag{2.15}$$

where $A^\kappa_{\mu\lambda}$ and $B^\kappa_{\mu\lambda}$ serve as the spin gauge fields in the ordinary theory of gauge fields (Yang and Mills, 1954; Utiyama, 1956), which are introduced in order to accomplish invariance under the (internal) Lorentz transformation in the spin space (or the four-dimensional Minkowski space) (Kibble, 1961). From this viewpoint, the coefficients of connection $\Gamma^{\kappa}_{\mu\lambda}$ and $C^{\kappa}_{\mu\lambda}$ may also be regarded as the two kinds of gauge fields which are introduced to achieve general covariance under the coordinate transformation (Bregman, 1973; Isham *et al.*, 1971; Kibble, 1961). Next, the covariant derivatives of α_λ are obtained from (2.8), (2.10), and (2.15) as follows {cf. (4.2) in [I]}:

$$\begin{aligned} \nabla_\mu \alpha_\lambda &= \overset{*}{\partial}_\mu \alpha_\lambda - \overset{*}{\Gamma}{}^{\kappa}_{\mu\lambda}\alpha_\kappa - \overset{*}{A}{}^{\kappa}_{\mu\lambda}\alpha_\kappa \qquad (=0) \\ \nabla_{\dot\mu}\alpha_\lambda &= \partial_{\dot\mu}\alpha_\lambda - C^{\kappa}_{\mu\lambda}\alpha_\kappa - B^{\kappa}_{\mu\lambda}\alpha_\kappa \qquad (=0) \end{aligned} \tag{2.16}$$

where

$$\overset{*}{A}{}^{\kappa}_{\mu\lambda} = A^{\kappa}_{\mu\lambda} - G_\mu{}^\nu B^{\kappa}_{\nu\lambda} \tag{2.17}$$

As to ψ, since ψ also depends on x and ρ in this case, its absolute differential is defined by [cf. (2.3) and (2.8)]

$$\begin{aligned} D\psi &= d\psi - \Gamma_\mu \psi dx^\mu - C_\mu \psi d\rho^\mu \\ &= d\psi - \overset{*}{\Gamma}_\mu \psi dx^\mu - C_\mu \psi \delta\rho^\mu \end{aligned} \tag{2.18}$$

where Γ_μ and C_μ mean the spin affine connections (Takano, 1968; Utiyama, 1956), and we have put

$$\overset{*}{\Gamma}_\mu = \Gamma_\mu - G_\mu{}^\nu C_\nu \tag{2.19}$$

Then, two kinds of covariant derivatives of ψ are defined as follows {cf. (4.3) in [I]}:

$$\begin{aligned} \nabla_\mu \psi &= \overset{*}{\partial}_\mu \psi - \overset{*}{\Gamma}_\mu \psi \\ \nabla_{\dot\mu}\psi &= \partial_{\dot\mu}\psi - C_\mu \psi \end{aligned} \tag{2.20}$$

By the way, $\overset{*}{\Gamma}_\mu$ and C_μ are related to $\overset{*}{A}{}^{\kappa}_{\mu\lambda}$ and $B^{\kappa}_{\mu\lambda}$, respectively, as follows:

$$\begin{aligned} \overset{*}{\Gamma}_\mu \alpha_\lambda - \alpha_\lambda \overset{*}{\Gamma}_\mu &= \overset{*}{A}{}^{\kappa}_{\mu\lambda}\alpha_\kappa \qquad (= \overset{*}{\partial}_\mu \alpha_\lambda - \overset{*}{\Gamma}{}^{\kappa}_{\mu\lambda}\alpha_\kappa) \\ C_\mu \alpha_\lambda - \alpha_\lambda C_\mu &= B^{\kappa}_{\mu\lambda}\alpha_\kappa \qquad (= \partial_{\dot\mu}\alpha_\lambda - C^{\kappa}_{\mu\lambda}\alpha_\kappa) \end{aligned} \tag{2.21}$$

from which $\overset{*}{\Gamma}_\mu$ and C_μ are determined, neglecting arbitrariness, as follows {cf. (4.4) in [I]} (Bregman, 1973; Takano, 1968):

$$\begin{aligned} \Gamma_\mu &= \tfrac{1}{4}\overset{*}{A}_{\mu\lambda\kappa} n^{\lambda\kappa} \\ C_\mu &= \tfrac{1}{4} B_{\mu\lambda\kappa} n^{\lambda\kappa} \end{aligned} \tag{2.22}$$

where $n^{\lambda\kappa} = \alpha^{[\lambda}\alpha^{\kappa]} = (\alpha^\lambda\alpha^\kappa - \alpha^\kappa\alpha^\lambda)/2$, which may be regarded as the Lorentz spin matrix, or the generator of the Lorentz group from another viewpoint (Bregman, 1973; Isham *et al.*, 1971; Kibble, 1961; Utiyama, 1956). On the other hand, three kinds of spin curvature tensors can be obtained by calculating such commutation relations as $\nabla_{[\nu}\nabla_{\mu]}\psi$, $\nabla_{[\nu}\nabla_{\dot\mu]}\psi$, and $\nabla_{[\dot\nu}\nabla_{\dot\mu]}\psi$

and the other six kinds of curvature tensors are also introduced through such commutation relations as $\nabla_{[\nu}\nabla_{\mu]}\alpha_{\lambda}$, $\nabla_{[\nu}\nabla_{\dot{\mu}]}\alpha_{\lambda}$, and $\nabla_{[\dot{\nu}}\nabla_{\dot{\mu}]}\alpha_{\lambda}$, which, however, are omitted here {cf. (4.7) and (4.10) in [I]}.

Now, we shall pay attention to $C_{\mu\lambda\kappa}$. In our case, as already mentioned, $C_{\mu\lambda\kappa}$ is assumed to be symmetrical with respect to all indices [cf. (2.5)] and the relations (2.6) and (2.7) are also assumed. In the $[x, \rho]$ field, this is related to the metric tensor $\alpha_{\lambda\kappa}$ $(\equiv \alpha_{(\lambda}\alpha_{\kappa)})$ through the metric condition $\nabla_{\dot{\mu}}\alpha_{\lambda\kappa} = 0$ as follows [cf. (2.6)]:

$$C_{\mu\lambda\kappa} = \partial_{\dot{\mu}}\alpha_{\lambda\kappa}/2 = \delta_{\mu(\lambda}\alpha_{\kappa)} \tag{2.23}$$

from which $C^{\lambda}_{\mu\lambda}$ is obtained, by contraction of the indices κ and λ, as follows:

$$C^{\lambda}_{\mu\lambda} = \alpha_{\mu} \tag{2.24}$$

On the other hand, since $C_{\mu\lambda\kappa}$ represents the deviation from Riemann space [i.e., the (x) field in our case] caused essentially by the electromagnetic potential (ρ), Horváth (1950) proposed physically the following relation:

$$C_{\mu\lambda\kappa} = -\tfrac{1}{2}(\nabla_{\mu}S_{\lambda\kappa} + \nabla_{\lambda}S_{\kappa\mu} + \nabla_{\kappa}S_{\mu\lambda}) \tag{2.25}$$

where $S_{\lambda\kappa}$ denotes the symmetrized stress-energy tensor. Therefore, from (2.25), $C^{\lambda}_{\mu\lambda}$ is given by, by virtue of $S_{\lambda}^{\cdot\lambda} = 0$,

$$C^{\lambda}_{\mu\lambda} = -\nabla_{\lambda}S_{\mu}^{\cdot\lambda} \tag{2.26}$$

Then, comparing (2.24) with (2.26), we can put

$$\alpha_{\mu} = -\nabla_{\lambda}S_{\mu}^{\cdot\lambda} \tag{2.27}$$

which represents the density of the Lorentz force (Horváth, 1950; Horváth und Moor, 1952).

Finally, we shall obtain the fundamental equation for ψ by taking account of the parallel displacement of the metric $dz\psi$. This procedure has been used in wave geometry (Mimura & Takeno, 1962; Mimura *et al.*, 1967). In our case, since the electromagnetic potential (ρ) (i.e., Weyl's gauge field) is already fused into the spatial structure of the (x, ρ) field, this parallel displacement implies a generalization of Weyl's gauge invariance with respect to the metric $(dz\psi)$. Now, if the vector ξ^{λ} which constitutes the metric (or the gauge) by $\alpha_{\lambda}\xi^{\lambda}\psi$ at the point (x) is displaced in parallel from the point (x) to its neighboring point $(x + dx)$, then the change of $(\alpha_{\lambda}\xi^{\lambda}\psi)$ is assumed to be given by

$$d(\alpha_{\lambda}\xi^{\lambda}\psi) = (\Sigma_{\mu}dx^{\mu} + \Sigma_{\dot{\mu}}\delta\rho^{\mu})(\alpha_{\lambda}\xi^{\lambda}\psi) \tag{2.28}$$

where Σ_{μ} and $\Sigma_{\dot{\mu}}$ mean arbitrary 4 x 4 matrices {cf. (3.3) and (3.4) in [I]}. Then, taking account of (2.8), (2.15), and (2.16), we can obtain two kinds of field equations in the $[x, \rho]$ field as follows:

$$\begin{aligned} \alpha_{\lambda}(\overset{*}{\partial}_{\mu}\psi) + (\overset{*}{\partial}_{\mu}\alpha_{\lambda})\psi - (\alpha_{\kappa}\overset{*}{\Gamma}{}^{\kappa}_{\mu\lambda})\psi &= \Sigma_{\mu}\alpha_{\lambda}\psi \\ \alpha_{\lambda}(\partial_{\dot{\mu}}\psi) + (\partial_{\dot{\mu}}\alpha_{\lambda})\psi - (\alpha_{\kappa}C^{\kappa}_{\mu\lambda})\psi &= \Sigma_{\dot{\mu}}\alpha_{\lambda}\psi \end{aligned} \tag{2.29}$$

which, by use of (2.16), (2.20), and (2.21), are also rewritten as

$$\begin{aligned} \alpha_\lambda \nabla_\mu \psi + \overset{*}{\Gamma}_\mu \alpha_\lambda \psi &= \Sigma_\mu \alpha_\lambda \psi \\ \alpha_\lambda \nabla_{\dot{\mu}} \psi + C_\mu \alpha_\lambda \psi &= \Sigma_{\dot{\mu}} \alpha_\lambda \psi \end{aligned} \tag{2.30}$$

Here, if we multiply (2.30) by α_λ and divide it by $\alpha_{\lambda\lambda}$, where we do not sum for λ, then we can obtain the fundamental equations for ψ in such formulas as $\nabla_\mu \psi = \Sigma'_\mu \psi$ and $\nabla_{\dot{\mu}} \psi = \Sigma'_{\dot{\mu}} \psi$, where Σ'_μ and $\Sigma'_{\dot{\mu}}$ stand for arbitrary 4×4 matrices. These equations have been adopted in wave geometry. On the other hand, if we contract (2.30) over the indices λ and μ, we can obtain the fundamental equations of the form $\alpha^\mu \nabla_\mu \psi - \kappa\psi = 0$ and $\alpha^\mu \nabla_{\dot{\mu}} \psi - \lambda\psi = 0$, where κ and λ are certain constants. These equations should be regarded as the generalized Dirac equations (Ikeda, 1974; Takano, 1968; Yukawa, 1950).

At any rate, we may say that this kind of Finslerian nonlocal field can bring out the "anisotropy" of the physical field caused by the internal variable in full relief (Horváth, 1958). And from this viewpoint, it may be expected that this "anisotropy" can be related to some kinds of broken symmetries within the scope of the theory of gauge fields (Bregman, 1973; Horváth, 1958; Isham *et al.*, 1971; Kibble, 1961; Utiyama, 1956; Yang and Mills, 1954). Therefore, in future, we should also consider this problem by focusing particular attention on the spin gauge fields and some "interactions" between the (x, ρ) and (ψ) fields.

3. *On a "Nonlocalization" of the* (x, ψ) *Field*

In this section, we shall consider a generalization of the (x, ψ) field by taking the spinor (ψ) as an independent variable. Then, both the point x and the spinor ψ are taken as the independent variables in this generalized nonlocal field (i.e., the $[x, \psi]$ field). And as already mentioned in section 1, the $[x, \psi]$ field cannot be grasped by Finsler geometry, so that some generalization of the (x, ρ) field treated in the previous section must be attempted, as will be done in the following.

First, since the metric in the $[x, \psi]$ field should embody the interaction between the (x) and (ψ) fields, it should be given in the same form as (2.11). However, in this case, it is desirable to show explicitly that the metric depends on x and ψ, so that we choose $G_\lambda \equiv g_\lambda \psi$ as the matrix metric which can play the same role as g_λ and α_λ do. Then, the metric tensor in the $[x, \psi]$ field is defined by $G_{\lambda\kappa} = G_{(\lambda}G_{\kappa)}$.

Next, by taking dx^μ and $d\psi$ as the independent (line) elements, the absolute differential of an arbitrary vector X^κ is given by

$$DX^\kappa = dX^\kappa + \Xi^\kappa_{\mu\lambda} X^\lambda dx^\mu + H_\lambda{}^\kappa X^\lambda d\psi \tag{3.1}$$

where $\Xi^\kappa_{\mu\lambda}$ and $H_\lambda{}^\kappa$ denote the coefficients of connection in the $[x, \psi]$ field [cf. (2.3)]. As to ψ, its intrinsic absolute differential should be given by the base connection (i.e., $\delta\psi$) from a purely geometrical viewpoint, and in this case, $\delta\psi$ cannot be identified with $D\psi$ (Kawaguchi, 1962). Therefore, in general, we shall define $\delta\psi$ in the form

$$\delta\psi = d\psi - \Theta_\mu \psi dx^\mu - C\psi d\psi \tag{3.2}$$

where Θ_μ and C mean the spin affine connections [cf. (2.18)]. Here, for the sake of simplicity, we shall also assume the homogeneity condition about C, i.e., $C\psi = 0$ corresponding to (2.7). Then, (3.2) is reduced to

$$\delta\psi = d\psi - \Theta_\mu \psi dx^\mu = (\nabla_\mu \psi) dx^\mu \tag{3.3}$$

where the covariant derivative of ψ (i.e., $\nabla_\mu \psi$) is defined by [cf. (2.20)]

$$\nabla_\mu \psi = \partial_\mu \psi - \Theta_\mu \psi \tag{3.4}$$

Now, substituting (3.3) into (3.1), we can rewrite (3.1) as

$$\begin{aligned} DX^\kappa &= dX^\kappa + \bar{\Xi}^\kappa_{\mu\lambda} X^\lambda dx^\mu + H_\lambda{}^\kappa X^\lambda \delta\psi \\ &= (\nabla_\mu X^\kappa)\, dx^\mu + (\nabla X^\kappa)\delta\psi \end{aligned} \tag{3.5}$$

where

$$\bar{\Xi}^\kappa_{\mu\lambda} = \Xi^\kappa_{\mu\lambda} + H_\lambda{}^\kappa (\Theta_\mu \psi) \tag{3.6}$$

[cf. (2.9)], and the covariant derivatives of X^κ are defined by

$$\nabla_\mu X^\kappa = \bar{\partial}_\mu X^\kappa + \bar{\Xi}^\kappa_{\mu\lambda} X^\lambda, \qquad \nabla X^\kappa = \partial X^\kappa + H_\lambda{}^\kappa X^\lambda \qquad \bar{\partial}_\mu \equiv \partial_\mu + (\Theta_\mu \psi)\partial, \tag{3.7}$$

where ∂ ($\equiv \partial/\partial\psi$) means the partial differentiation with respect to a component of the spinor (ψ) [cf. (2.10)].

Now, we shall proceed to the connection of G_λ. Its absolute differential should be given by

$$DG_\lambda = dG_\lambda - \bar{\Xi}^\kappa_{\mu\lambda} G_\kappa dx^\mu - M^\kappa_{\mu\lambda} G_\kappa dx^\mu - H_\lambda{}^\kappa G_\kappa d\psi - N_\lambda{}^\kappa G_\kappa d\psi \tag{3.8}$$

where $M^\kappa_{\mu\lambda}$ and $N_\lambda{}^\kappa$ mean the spin gauge fields [cf. (2.15)], which may be related to the spin affine connections Θ_μ and C, respectively, as follows [cf. (2.21)]:

$$\begin{aligned} M^\kappa_{\mu\lambda} G_\kappa &= \Theta_\mu G_\lambda - G_\lambda \Theta_\mu \\ N_\lambda{}^\kappa G_\kappa &= C G_\lambda - G_\lambda C \end{aligned} \tag{3.9}$$

In the same way as (2.16), the covariant derivatives of G_λ are defined by

$$\begin{aligned} \nabla_\mu G_\lambda &= \bar{\partial}_\mu G_\lambda - \bar{\Xi}^\kappa_{\mu\lambda} G_\kappa - \bar{M}^\kappa_{\mu\lambda} G_\kappa \qquad (=0) \\ \nabla G_\lambda &= \partial G_\lambda - H_\lambda{}^\kappa G_\kappa - N_\lambda{}^\kappa G_\kappa \qquad (=0) \end{aligned} \tag{3.10}$$

where we have put

$$\bar{M}^\kappa_{\mu\lambda} = M^\kappa_{\mu\lambda} + N_\lambda{}^\kappa (\Theta_\mu \psi) \tag{3.11}$$

[cf. (2.17)]. Of course, we may put $\nabla_\mu G_{\lambda\kappa} = 0$ and $\nabla G_{\lambda\kappa} = 0$. At this stage, if such relations as $\nabla_\mu G_{\lambda\kappa} = \phi_\mu G_{\lambda\kappa}$ and $\nabla G_{\lambda\kappa} = \phi G_{\lambda\kappa}$ are taken into account, then two kinds of gauge fields in Weyl's style can be introduced into the $[x, \psi]$ field (Ikeda, 1974; Weyl, 1918). By the way, if the base field, i.e., the (x) field is generalized to a Finslerian field by taking further the electromagnetic potential (ρ) as an independent variable, as has been done in the previous section, then (3.1) is extended to

$$DX^\kappa = dX^\kappa + \Xi^\kappa_{\mu\lambda} X^\lambda\, dx^\mu + E^\kappa_{\mu\lambda} X^\lambda d\rho^\mu + H_\lambda{}^\kappa X^\lambda d\psi \tag{3.12}$$

where $E^\kappa_{\mu\lambda}$ is the coefficient of connection coping with $C^\kappa_{\mu\lambda}$ introduced by (2.3). Then, for the sake of simplicity, assuming the same homogeneity condition as (2.7) and $H_\lambda{}^\kappa \rho^\lambda = 0$ and identifying the base connection of ρ (i.e., $\delta\rho$) with $D\rho$ derived from (3.12), we can obtain, for example, the base connection of ψ [cf. (3.3)] and the covariant derivatives of ψ [cf. (3.4)], respectively, as follows [cf. (2.18) and (2.20)]:

$$\begin{aligned} \delta\psi &= d\psi - \Theta_\mu \psi\, dx^\mu - \Omega_\mu \psi d\rho^\mu \\ &= d\psi - \overset{*}{\Theta}_\mu \psi\, dx^\mu - \Omega_\mu \psi \delta\rho^\mu \end{aligned} \tag{3.13}$$

where Ω_μ denotes another spin affine connection coping with C_μ and $\overset{*}{\Theta}_\mu = \Theta_\mu - \Xi^\nu_{\mu\lambda}\rho^\lambda(\Omega_\nu\psi)$, and

$$\begin{aligned} \nabla_\mu \psi &= \partial_\mu \psi - \Xi^\nu_{\mu\lambda}\rho^\lambda \partial_{\dot\nu}\psi - \overset{*}{\Theta}_\mu \psi \\ \nabla_{\dot\nu}\psi &= \partial_{\dot\mu}\psi - \Omega_\mu \psi \end{aligned} \tag{3.14}$$

Finally, we shall obtain the fundamental equation for ψ in the $[x, \psi]$ field with the aid of the parallel displacement of the metric $(G_\lambda\, dx^\lambda)$. Now, when the vector ξ constituting the metric by $G_\lambda \xi^\lambda$ at the point (x) is displaced in parallel from the point (x) to its neighboring point $(x + dx)$, the change of $(G_\lambda \xi^\lambda)$ is assumed to be given by

$$d(G_\lambda \xi^\lambda) = (\Pi_\mu\, dx^\mu)(G_\lambda \xi^\lambda) \tag{3.15}$$

where Π_μ denotes an arbitrary 4×4 matrix [cf. (2.28)]. Of course, (3.15) represents a generalization of Weyl's gauge invariance (Weyl, 1918). Calculating the left-hand side of (3.15) with the use of (3.3), (3.5), (3.8), and (3.10), we can obtain from (3.15)

$$\overline{M}^\kappa_{\mu\lambda} G_\kappa + (N_\lambda{}^\kappa G_\kappa)(\nabla_\mu \psi) = \Pi_\mu G_\lambda$$

or (3.16)

$$\overline{M}^\kappa_{\mu\lambda} g_\kappa + (N_\lambda{}^\kappa g_\kappa)(\nabla_\mu \psi) = \Pi_\mu g_\lambda$$

Then, after some manipulation, this equation can be rectified in the form

$$g_\kappa \nabla_\mu \psi = \Lambda_\mu g_\kappa \tag{3.17}$$

where Λ_μ stands for an arbitrary 1×4 matrix which depends suitably on g_λ, C, and Θ_μ. Therefore, from (3.17), two different types of the fundamental equation for ψ, i.e., $\nabla_\mu \psi = \Lambda'_\mu$ $(\equiv g_\kappa \Lambda_\mu g_\kappa / g_{(\kappa} g_{\kappa)})$ and $g^\mu \nabla_\mu \psi = \Lambda$ $(\equiv \Lambda_\mu g^\mu)$ can be obtained, as has already been mentioned in section 2.

Thus, the spatial structure of the $[x, \psi]$ field has been made clear to some extent and some inherent characteristics of the spinor ψ have been fused into the spatial structure. Therefore, in future, we should investigate some physical features underlying this generalized nonlocal field by making reference to the nonlocal field theory (Yukawa, 1950; Takano, 1968) and the Finslerian unified field theory (Horváth, 1950; Horváth und Moór, 1952; Lichnerowicz, 1955; Tonnelat, 1965, 1966).

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